

The arithmetical rank of a special class of monomial ideals

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Given a commutative noetherian ring with identity R , the arithmetical rank of an ideal I of R , denoted by $\text{ara } I$, is defined as the minimum number of elements that generate an ideal having the same radical as I . When I is a square-free monomial ideal, then Lyubeznik [3] proved that

$$\text{pd}(R/I) \leq \text{ara } I, \quad (1)$$

where $\text{pd}(R/I)$ denotes the projective dimension of the module R/I . In this paper we want to generalize result in [2], Theorem 4.5, verifying that for all t, n positive integers the equality

$$\text{ara}(I_t(L_n)) = \text{pd}(I_t(L_n)) \quad (2)$$

holds, where

$$I_t(L_n) = \{x_i x_{i+1} \cdots x_{i+t-1} \mid i = 1, \dots, n - t + 1\}.$$

To prove 2, we need the following lemma.

Lemma 1. *Let I, J be monomial ideals in the ring $K[x_1, \dots, x_N]$. Suppose that each generator of J is divisible by an indeterminate x_i (not necessary the same for all monomials) such that no generators of I is multiple of x_i . Then $\text{ara } I \leq \text{ara}(I + J)$.*

Proof. Without loss of generality, we can assume that all monomials generating I, J are square-free, thus I, J are radical ideals. We can also assume that I is contained in $K[x_1, \dots, x_r]$ for some r and that all monomials of J are divisible by some x_j with $j > r$. Let $n = \text{ara}(I + J)$ and let g_1, \dots, g_n be elements such that $\sqrt{(g_1, \dots, g_n)} = (I + J)$. For all $i = 1, \dots, n$ we set f_i equals to g_i without monomials divisible by x_j for some $j > r$. We now show that $\sqrt{(f_1, \dots, f_n)} = I$.

Since I, J are monomial ideals, obviously $f_i \in I$ for all $i \leq n$. Let h be an element of I , then $h \in I + J$, so $h^m \in (g_1, \dots, g_n)$ for some integer m . Then,

$$h^m = \sum_{i=1}^n a_i g_i \text{ with } a_i \in K[x_1, \dots, x_n]. \text{ We know that } h \in K[x_1, \dots, x_r] \text{ so every linear combination with indeterminates } x_j, j > r \text{ must be 0. Then } h^m = \sum_{i=1}^n b_i f_i \text{ where each } b_i \text{ is obtained by } a_i, \text{ deleting some monomials.}$$

Then $h \in \sqrt{(f_1, \dots, f_n)}$. □

Note that in general inequality $\text{ara } I \leq \text{ara}(I + J)$ is not true. In fact, let $I = (x_1 x_2, x_1 x_3, x_4 x_5), J = (x_1)$ in $K[x_1, \dots, x_5]$. I has (x_2, x_3, x_4) as minimal prime ideal; this prime ideal has height 3 and is known that this is a lower bound for the arithmetical rank of I . So $\text{ara } I = 3$, but $\text{ara}(I + J) = 2$ since $I + J = (x_1, x_4 x_5)$.

We now present a result due to Barile [1].

Proposition 1. *For all $n \geq 1$ let I_n be the ideal of $K[x_1, \dots, x_{2n}]$ generated by the following monomials*

$$A = \{x_1 \cdots x_n, x_2 \cdots x_{n+1}, \dots, x_{n+1} \cdots x_{2n}\}.$$

Then, $\text{ara } A = 2$.

Proof. [1], Proposition 3.1. □

Now we can give a proof of (2).

Corollary 1.

$$\text{ara}(R/I_t(L_n)) = \begin{cases} \frac{2(n-d)}{t+1} & \text{if } n \equiv d \pmod{t+1} \text{ with } 0 \leq d \leq t-1 \\ \frac{2n-(t-1)}{t+1} & \text{if } n \equiv t \pmod{t+1}. \end{cases}$$

Proof. We can write $n = k(t+1) + d$ with $0 \leq d \leq t$. We know that $I_t(L_n)$ consists of $n - t + 1 = k(t+1) + d - t + 1$ monomials of the form $x_i \dots x_{i+t-1}$. By previous proposition, the radical of the ideal is the same if we substitute $t+1$ consecutive monomial by two opportune polynomials. For this reason, we subdivide these monomials in sets of $t+1$ elements. First we consider the case $d = t - 1$. Then we get exactly k sets, so by previous proposition $\text{ara } I_t(L_n) \leq 2k = \frac{2(n-d)}{t+1}$.

Now suppose $d < t - 1$. Let J be the ideal generated by monomials $x_i \dots x_{i+t-1}$ for $i = k(t+1) + d - t + 2, \dots, k(t+1)$. The ideal $I_t(L_n) + J$ is in the case $d = t - 1$ so $\text{ara}(I_t(L_n) + J) \leq 2k$. By Lemma 1 we get $\text{ara } I_t(L_n) \leq 2k = \frac{2(n-d)}{t+1}$.

Finally suppose $d = t$, then there are $k(t+1) + 1$ monomials. As in the previous cases, we can replace the first $k(t+1)$ elements with $2k$ polynomials, without change the radical of the ideal, so we get $2k + 1$ elements. Therefore $\text{ara } I_t(L_n) \leq 2k + 1 = \frac{2n-(t-1)}{t+1}$.

The claim follows immediately by projective dimension of $R/I_t(L_n)$ computed in [2], Theorem 4.1 and by 1. \square

References

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